Lecture 12

Covariant Formulation of CED.

- Recap (Lorentz transformations, 4-vectors)
  [Add Poincaré group]
- Addition of velocities, interval
- Transformation of Φ and $\vec{A}$, A^μ
  Lorentz inv. and Maxwell equations
- Field strength tensor $F_{\mu\nu}$, gauge inv.
- Covariant form of Maxwell equations

Recap:

$$\eta_{\mu\nu} = \eta^{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & 0 \\ & 0 & 1 & \\ & & & 1 \end{pmatrix}$$

$$x_\mu = \eta_{\mu\nu} x^\nu = (-x^0, x^1, x^2, x^3)$$

$$x^0 = ct$$

$$\left.\begin{aligned} t' &= t\cosh\chi + \frac{x_1}{c}\sinh\chi \\ x_1' &= x_1\cosh\chi + ct\sinh\chi \end{aligned}\right\} \text{Lorentz Boost}$$

→ change notation

$$x^{\mu'} = \Lambda^\mu{}_\nu x^\nu \quad , \quad x_\mu' = \Lambda_\mu{}^\nu x_\nu$$

$$\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma \eta^{\rho\sigma} = \eta^{\mu\nu'} = \eta^{\mu\nu} \rightarrow$$

→ Invariant

Poincaré : translations + Lorentz

$$\hat{\Lambda} = \hat{\Lambda}_{01} \hat{\Lambda}_{02} \hat{\Lambda}_{03} \hat{\Lambda}_{12} \hat{\Lambda}_{13} \hat{\Lambda}_{23}$$

(order matters)

Objects with contracted indices invariant
(cov. vs inv.)

we treat $\frac{\partial}{\partial x^\mu}$ as an object with lower index: ∂_μ.

$$\Box = -\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu}\eta^{\mu\nu} \rightarrow \text{Lorentz inv.}$$

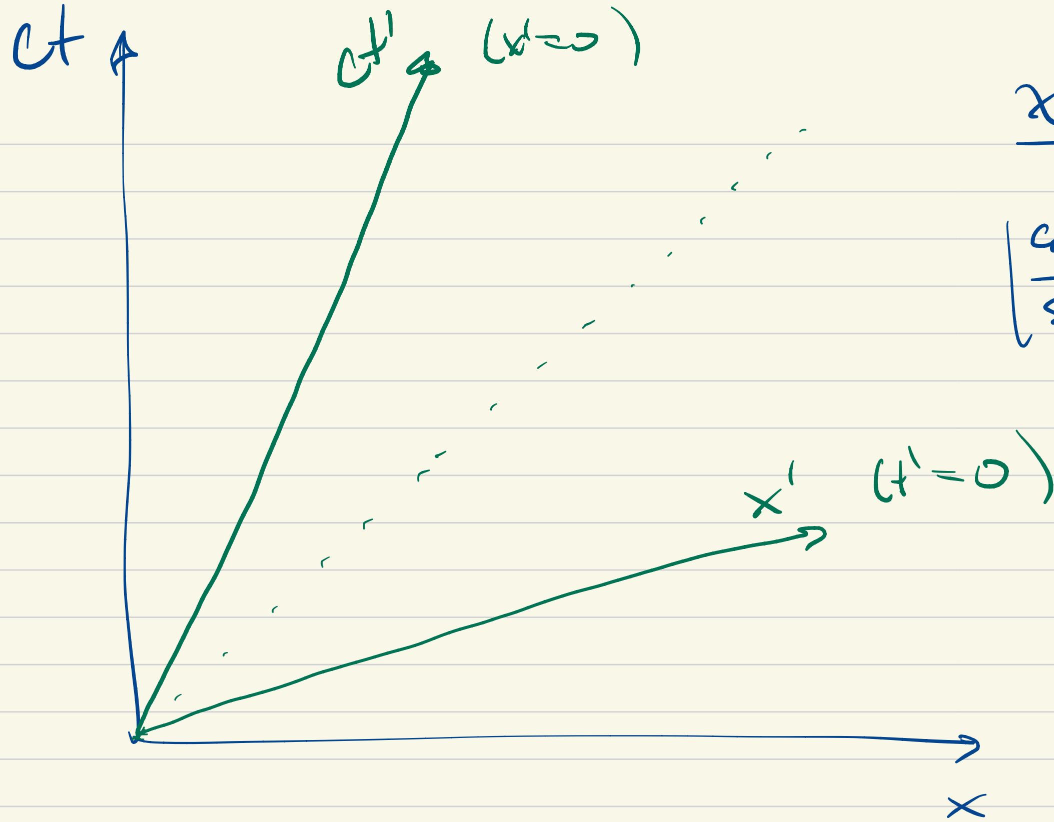
Lorentz transform : $x^{\mu\prime} = \Lambda^\mu{}_\nu x^\nu$

Addition of velocities

$$\begin{cases} ct' = ct\cosh\chi + x_1 \sinh\chi \\ x_1' = x_1 \cosh\chi + ct \sinh\chi \\ x_2' = x_2 \\ x_3' = x_3 \end{cases}$$

$$x' = 0 \Rightarrow ct = -x\frac{\cosh\chi}{\sinh\chi}$$

$$t' = 0 \Rightarrow ct = -x\frac{\sinh\chi}{\cosh\chi}$$



$\chi < 0$

$$\left|\frac{\cosh\chi}{\sinh\chi}\right| \geq 1$$

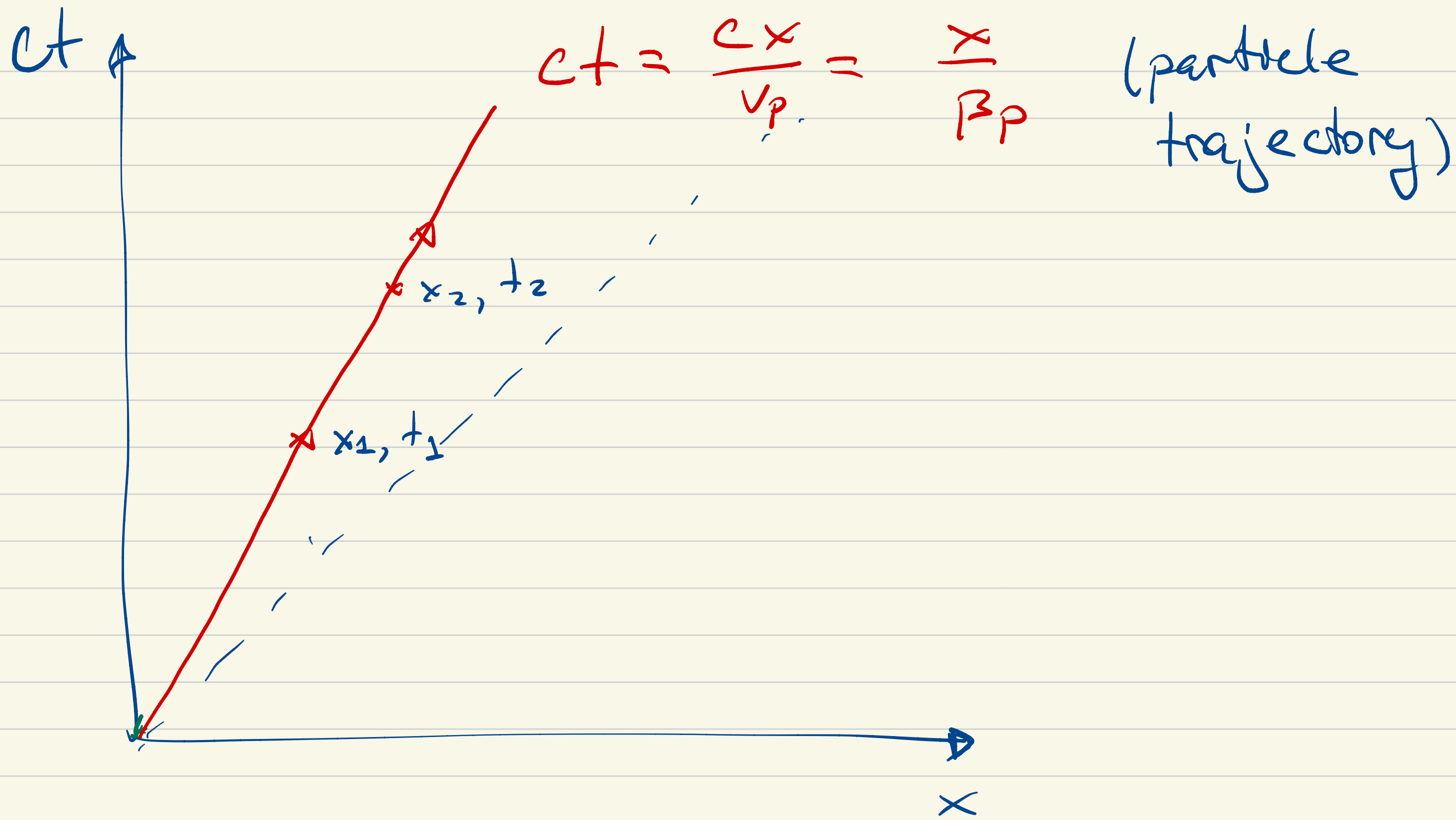
$$x' = 0 \Rightarrow \quad x = -ct\,\frac{\sinh\chi}{\cosh\chi} \equiv vt$$

$$\frac{v}{c} \equiv -\frac{\sinh\chi}{\cosh\chi} \equiv \beta$$ [changed notation]

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$ [boost] $\gamma \geq 1$

$$ct' = \gamma(ct - \beta x)$$

$$x' = \gamma(x - \beta ct)$$



what is the velocity of the particle in the new reference frame?

$x_2, t_2 \longrightarrow x_2', t_2'$

$x_1, t_2 \longrightarrow x_1', t_1'$

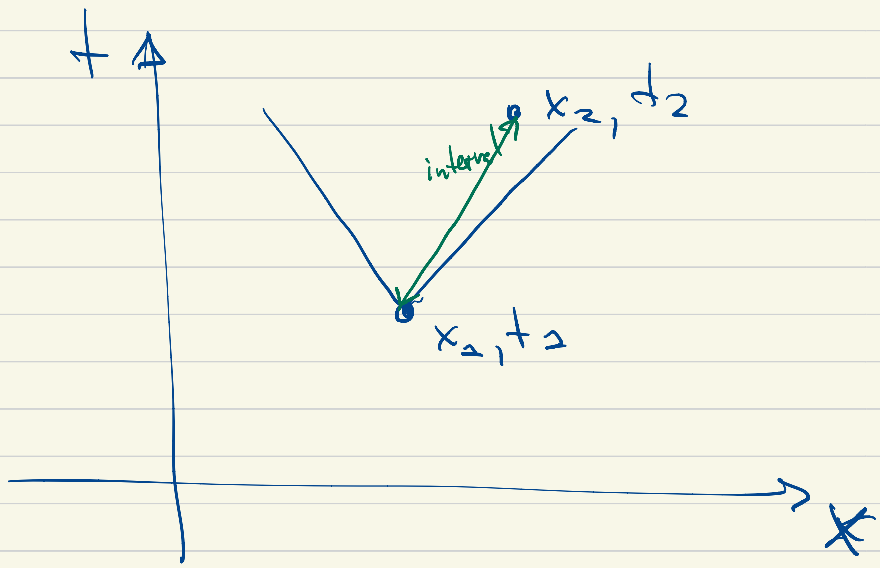
$$\frac{v_p'}{c} = \frac{1}{c}\frac{x_2' - x_1'}{(t_2' - t_1')} = \frac{\gamma((x_2 - x_1) - \beta c(t_2 - t_1))}{\gamma((ct_2 - ct_1) - \beta(x_2 - x_1))}$$

$$= \frac{v_p - \beta c}{c - \beta v_p} \Rightarrow v_p' = \frac{v_p - v}{1 - \frac{v v_p}{c^2}}$$

$$v_p = c \Rightarrow v_p' = c \,!$$

Interval

$$s^2 = -c^2 \Delta t^2 + (\Delta \vec{x})^2 = \Delta x^\mu \Delta x_\mu$$



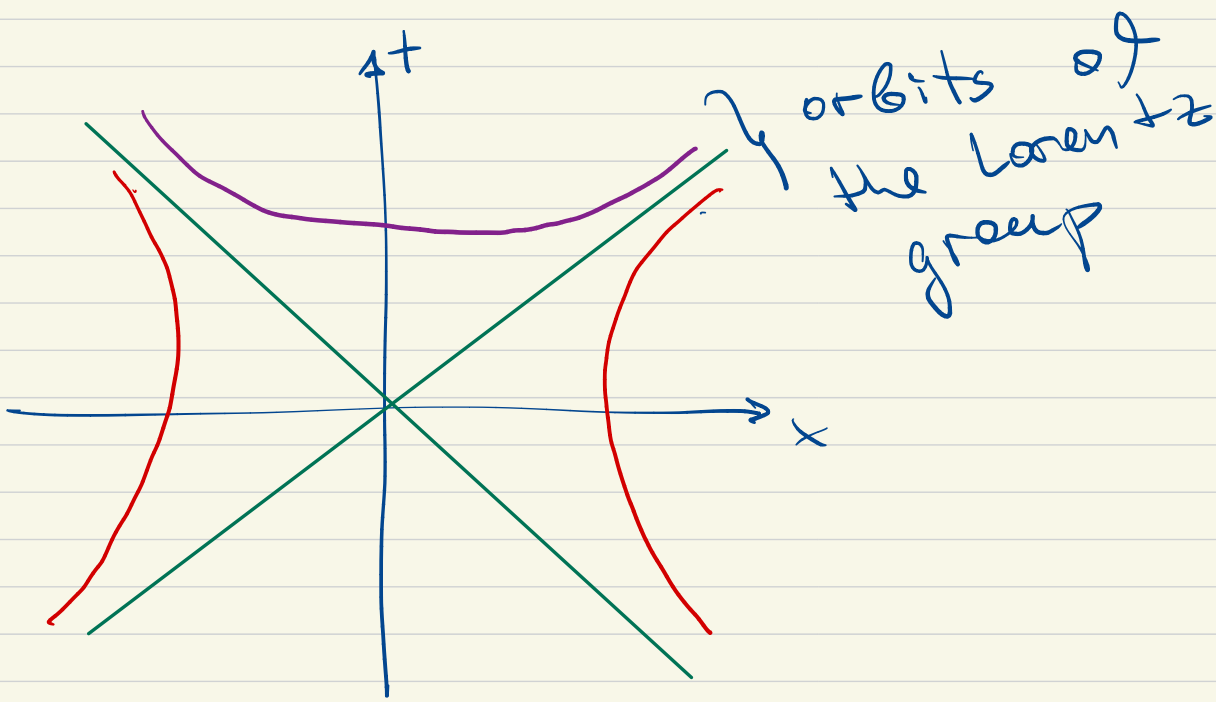
- interval corresponds to the "internal" or proper time of the object

- Time inside the objects moving with some speed seems to be slowed down when observed from the outside

$s^2 < 0 \Rightarrow$ time-like interval (signal can reach)

$s^2 > 0 \Rightarrow$ space-like interval (signal cannot reach)

$s^2 = 0 \Rightarrow$ null interval ~ lightcone



Transformation of Φ and $\vec{A}$, Lorentz inv. and Maxwell equations

First, let us study Galilean transformations of the charge density and current density: $t(t',\vec{x}'), \vec{x}(t',\vec{x}')$

$$\rho(t,\vec{x}) \to \rho'(t',\vec{x}') = \rho(\widetilde{t,\vec{x}})$$

$$\vec{J}(t,x) \to \vec{J}(t,x) + \vec{v}\rho(t,x)$$

this looks like a 4-vector transformation

$$t \to t' = t \qquad \vec{j} = q\vec{v}$$

$$\vec{x} \to \vec{x}' = \vec{x} + \vec{v}\cdot t$$

$$(c\rho, \vec{J}) \equiv J^\mu \to \text{4-vector-valued}$$

function: $\quad J^\mu(x^\mu)$

It is natural to expect that it transforms as a four-vector under full relativistic transformations:

$$J^\mu(x^\mu) \longrightarrow \Lambda^\mu{}_\rho J^\rho(x^\rho(x))$$

Charge conservation is Lorentz inv:

$$\partial_\mu J^\mu = \frac{c}{c}\partial_t \rho + \vec{\nabla}\cdot\vec{J} = 0$$

$$\begin{aligned} \Box \varphi &= \frac{1}{\varepsilon_0}\rho \\ \Box \vec{A} &= \mu_0 \vec{J} \end{aligned} \Rightarrow \Box \begin{pmatrix} \varphi \\ c\vec{A} \end{pmatrix} = \frac{1}{c\varepsilon_0}\begin{pmatrix} c\rho \\ \vec{J} \end{pmatrix}$$

To maintain Lorentz invariance $(\varphi, c\vec{A})$ should also transform as a 4-vector: $A^\mu(x^\mu)$

different people!

Lorenz / gauge condition is also Lorentz invariant:

$$\partial_\mu A^\mu = 0 \Rightarrow \frac{\partial_t \varphi}{c} + c\nabla_i A_i = 0$$

Gauge symmetry is Lorentz covariant:

$$A^\mu = (\varphi, c\vec{A})$$

$$A_\mu \to A_\mu + c\,\partial_\mu \alpha$$

$$\varphi^0 \to \varphi^0 - \frac{\partial}{\partial t}\alpha \,, \quad \vec{A} \to \vec{A} + \partial_i \alpha$$

So we demonstrated full Lorentz invariance of Maxwell equations!

Now we will derive the covariant form of Maxwell equations also in terms of $\vec{E}$ and $\vec{B}$ (it will be very nice!)

Field strength tensor $F_{\mu\nu}$, gauge invariance

So far we had:

A^{μ} – Lorentz covariant, but not gauge invariant

$\vec{E}$, $\vec{B}$ – gauge inv., but not Lorentz covariant

We will now introduce a new object, called field strength tensor [also known as Maxwell tensor] which makes both gauge invariance and Lorentz symmetry manifest.

It will be the most compact formulation

Before we do that let us discuss the distinction between gauge symmetry (invariance) and usual symmetry:

Usual symmetry (Lorentz, translation, reflection ...)

$\downarrow$

Transforms a solution of equations into a different solution [can be distinguished]

Gauge symmetry ($A_\mu \to A_\mu + \partial_\mu \alpha$)

$\downarrow$

Transforms a solution into physically identical, indistinguishable solution.

Back to field strength:

$$F_{\mu\nu} \stackrel{\text{def}}{=} \partial_\mu A_\nu - \partial_\nu A_\mu$$

$F_{\mu\nu}$ is a tensor [4x4 table of numbers]

it is antisymmetric:

$$F_{\mu\nu} = - F_{\nu\mu}$$

We can raise and lower indices with

$\eta^{\mu\nu}$:
$$F^{\mu\nu} = \eta^{\mu\rho} \eta^{\nu\gamma} F_{\rho\gamma}$$

$F_{\mu\nu}$ transforms under Lorentz as a Lorentz tensor:

$$F_{\mu\nu} \longrightarrow \Lambda_\mu{}^\rho \Lambda_\nu{}^\gamma F_{\rho\gamma}(x^\mu(x'))$$

$$x^\mu \rightarrow x^{\mu'} = \Lambda^\mu{}_\nu x^\nu$$

$F_{\mu\nu}$ is gauge invariant:

$$A_\mu \to A_\mu + \partial_\mu \alpha$$

$$F_{\mu\nu} \to \partial_\mu A_\nu + \partial_\mu \partial_\nu \alpha -$$
$$- \partial_\nu A_\mu - \partial_\nu \partial_\mu \alpha = F_{\mu\nu}$$

- Now let us find the relation between $F_{\mu\nu}$ and $\vec{E}$ and $\vec{B}$:

remember that

$$\vec{B} = \vec{\nabla} \times \vec{A} \quad , \quad \vec{E} = -\vec{\nabla} \varphi - \partial_t \vec{A}$$

$$A^\mu = (\varphi, c\vec{A}) \to A_\mu = (-\varphi, c\vec{A})$$

$$F_{0i} = (0, \dot{A}_1 + \partial_1 \varphi, \dot{A}_2 + \partial_2 \varphi,$$
$$\dot{A}_3 + \partial_3 \varphi) =$$
$$= (0, -E_x, -E_y, -E_z)$$

F_{ij} - ? Let's check

$$F_{21} = c\partial_2 A_1 - c\,\partial_1 A_2$$

$$B_3 = \varepsilon_{321}\partial_2 A_1 + \varepsilon_{312}\partial_1 A_2 =$$
$$= -\partial_2 A_1 + \partial_1 A_2 \Rightarrow$$

$F_{21} = -c\,B_3$. Check other components to see:

$$F_{\mu\nu} = \begin{array}{r} \mu=0 \rightarrow \\ \\ \\ \\ \end{array} \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ & 0 & cB_z & -cB_y \\ \text{Anti-sym.} & & 0 & cB_x \\ & & & 0 \end{pmatrix}$$

Covariant form of Maxwell equations

Let's remember Maxwell equations:

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}$$

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\vec{\nabla}\cdot\vec{B} = 0$$

Or in Lorentz gauge

$$\Box\varphi = \frac{\rho}{\varepsilon_0}$$

$$\Box A^i = \mu_0\vec{J}$$

First two equations take a very simple form:

$$\partial_\mu F^{\mu\nu} = -\frac{1}{c\varepsilon_0}J^\nu \qquad (*)$$

It's simplest to check in Lorentz gauge:

$$\partial_\mu\partial^\mu A^\nu - \underset{0}{\cancel{\partial_\mu\partial^\nu A^\mu}} = -\frac{1}{c\varepsilon_0}J^\nu$$

$$\Box A^\nu = \frac{1}{c\varepsilon_0}J^\nu \qquad \left(\Box = \frac{\partial^2}{c^2\partial t^2} - \Delta\right)$$

But (*) is gauge invariant, so it is true in every gauge!

Two other equations can be written as

$$\varepsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma} = 0$$

where $\varepsilon^{\mu\nu\rho\sigma}$ is a 4-dimensional analog of ε^{ijk} (fully anti-symmetric tensor), $\varepsilon^{0123} = 1$

Again, if we substitute

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ then obviously

$$\varepsilon^{\mu\nu\rho\sigma}(\partial_\nu\partial_\rho A_\sigma - \partial_\nu\partial_\sigma A_\rho) = 0$$

But it is a non-trivial constraint on $F_{\mu\nu}$ if we don't use A^μ.

$\varepsilon^{\mu\nu\rho\sigma}$ transforms like a tensor under Lorentz. One can check that it is actually invariant

$$\varepsilon^{\mu\nu\rho\sigma\,\prime} = \varepsilon^{\mu\nu\rho\sigma}$$